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PERIOD MAP OF SURFACES WITH $p_g=1$, $c_1^2=2$ AND $\pi_1=\mathbf{Z}/2\mathbf{Z}$

Dedicated to Professor Kazo TSUJI on the occasion of his sixties birthday

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Introduction

Certain surfaces of general type with $p_g=c_1^2=1$ were first constructed as a counter-example of the infinitesimal Torelli theorem ([Ki]). All those surfaces with these numerical invariants are descrived as weighted complete intersections ([C.1]), and for the period map of these surfaces the followings are known:

(0.1) The generic infinitesimal Torelli theorem holds ([C.1]).

(0.2) The period map has some positive dimensional fibres ([T.1], (U.1]).

(0.3) The phenomenon (0.2) is explained as an effect of an automorphism on the variation of Hodge structure ([U.2]).

(0.4) The infinitesimal Torelli theorem by means of the mixed Hodge structures on the complements of the canonical curves holds under the assumption that the canonical curves are ample and smooth ([U.3]).

For the period map of the surfaces with $p_g=1$ and $c_1^2=2$, we encounter a similar situation. Todorov constructed certain simply connected surfaces with these numerical invariants through which the period map has positive dimensional fibres ([T.2]). Catanese and Debarre described all those surfaces with these numerical invariants ([C.D]). The moduli space of these surfaces has two connected components according to $\pi_1=\{1\}$ and $\mathbf{Z}/2\mathbf{Z}$. Oliverio and Catanese proved the generic infinitesimal Torelli theorem for the cases $\pi_1=\mathbf{Z}/2\mathbf{Z}$ and $\pi_1=\{1\}$ respectively ([O], [C.2]).

In this paper, we examine a general viewpoint of an effect of an automorphism on a period map in [U.2] in some surfaces with $p_g=1$, $c_1^2=2$ and $\pi_1=\mathbf{Z}/2\mathbf{Z}$ (§2). We also prove the infinitesimal Torelli theorem by means of the mixed Hodge structures on the complements of the canonical curves for those surfaces with these numerical invariants under the assumption that the canonical curves are ample and smooth. The argument is analogous to that in [U.3] (§3). §1 is the preliminary of the following sections.

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We wish to remark here a general theory of period maps by means of the mixed Hodge structures on the complements $X-Y$ of normal crossing divisors Y. In this generalization, we can go on to some extent along the same way as "ordinary" period maps of [Gri]. Especially, we proved that the differential of the "generalized" period map coincides (up to $\bigoplus_p (-1)^p$) with the map

$$(0.5)\quad H^1(T_X(-\log Y)) \longrightarrow \bigoplus_{p+q=n} \mathrm{Hom}\,(H^p(\Omega_X^q(\log Y)),\, H^{p-1}(\Omega_X^{q+1}(\log Y)))$$

induced from contractions ([U.3]). Moreover, last April, a great advance was made by Griffiths in this direction. That is that he proved the injectivity of the map (0.5) for a sufficiently ample, smooth divisor Y in an arbitrary smooth, projective variety X. Combining this with a general result in [U.3], we can prove the "ordinary" infinitesimal Torelli theorem for the above Y. The last assertion was also proved indipendently by Green ([Gre]).

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1. Surfaces with $p_g=1$, $c_1^2=2$ and $\pi_1=\mathbf{Z}/2\mathbf{Z}$

Catanese and Debarre ([C.D]) gave a description of the surfaces with $p_g=1$ and $c_1^2=2$. In particular, they showed that (Theorem 2.8 in [C.D]):

(1.1) *Any canonical model of a surface X with $p_g=1$, $c_1^2=2$ and $\pi_1=\mathbf{Z}/2\mathbf{Z}$ occurs as the quotient $\tilde{X}/\langle\tilde{\tau}\rangle$ of a weighted complete intersection $\tilde{X}\subset\mathbf{P}=\mathbf{P}(1, 1, 1, 2, 2)$ with only rational double points as singularities, given by a pair of partially normalized equations:*

$$\begin{cases} f = z_3^2 + wz_4 f^{(1)}(x_1, x_2) + w^4 f^{(0)} + w^2 f^{(2)}(x_1, x_2) + f^{(4)}(x_1, x_2) \\ g = z_4^2 + wz_3 g^{(1)}(x_1, x_2) + w^4 g^{(0)} + w^2 g^{(2)}(x_1, x_2) + g^{(4)}(x_1, x_2) \end{cases}$$

where

$$\deg w = \deg x_1 = \deg x_2 = 1, \quad \deg z_3 = \deg z_4 = 2,$$

$f^{(i)}(x_1, x_2)$ and $g^{(i)}(x_1, x_2)$ are homogeneous polynomials of degree i ($i=0, 1, 2, 4$),

$f^{(4)}$ and $g^{(4)}$ do not have common factors,

$f^{(0)}$ and $g^{(0)}$ are not both zero,

$\tilde{\tau}$ is the involution on $\tilde{X}$, which induces a projectivity

$$(w, x_1, x_2, z_3, z_4) \longmapsto (w, -x_1, -x_2, -z_3, -z_4)$$

through the induced action on the canonical ring of $\tilde{X}$.

Moreover two such pairs of equations (f_1, g_1) *and* (f_2, g_2) *give rise to isomorphic surfaces if and only if there exists a projectivity* $h\colon \mathbf{P}\to\mathbf{P}$ *such that*

i) $h(w, x_1, x_2, z_3, z_4) = (cw, c_1x_1 + c_{12}x_2, c_{21}x_1 + c_2x_2, c_3z_3, c_4z_4)$

ii) *either* $c_3^2 f_2 = f_1\circ h$, $c_4^2 g_2 = g_1\circ h$

or $c_3^2 f_2 = g_1\circ h\circ i$, $c_4^2 g_2 = f_1\circ h\circ i$,

where $i\colon \mathbf{P}\to\mathbf{P}$ *is the involution permuting* z_3 *with* z_4.

Let X and $\tilde{X}$ be as in (1.1). We assume throughout this paper that *the unique canonical curve* C *of* X *is ample and smooth.* Then, in particular, the canonical model of X is smooth and we can identify X with its canonical model. Denote by $\tilde{C}$ the pull-back of C by the projection

$$p\colon \tilde{X} \longrightarrow X. \tag{1.2}$$

Let $L\in \mathrm{Pic}(X)$ be the 2-torsion corresponding to the double cover (1.2). Then we have:

$$\begin{aligned} &p^*\Omega^q_X(\log C) = \Omega^q_{\tilde{X}}(\log \tilde{C}). && p^*\Omega^q_X = \Omega^q_{\tilde{X}}. \\ &p^*T_X(-\log C) = T_{\tilde{X}}(-\log \tilde{C}). && p^*T_X = T_{\tilde{X}}. \\ &p_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X \oplus L^{-1}. \end{aligned} \tag{1.3}$$

Lemma 2.1 in [C.D] yields, in our case, that:

(1.4) *The canonical curve* C *of* X *is a smooth hyperelliptic curve of genus* 3.

Lemma (1.5) $H^0(T_X)=0$. $H^2(T_X)=0$. $\dim H^1(T_X)=16$.

Proof. Since $T_X \simeq \Omega^1_X\otimes K_X^{-1}$ and K_X is ample, we have $H^0(T_X)=0$ by the Kodaira-Nakano vanishing theorem. We get

$$H^2(T_{\tilde{X}}) = 0 \tag{1.6}$$

by a standard calculation of cohomology groups for a weighted complete intersection (cf. [M]), by using an exact diagram:

$$\begin{tikzcd} & & 0 & & \\ 0 \arrow[r] & T_{\tilde{X}} \arrow[r] & T_{\mathbf{P}}|\tilde{X} \arrow[r] \arrow[u] & N_{\tilde{X}/\mathbf{P}} \arrow[r] & 0 \\ & & \bigoplus_i \mathcal{O}_{\tilde{X}}(e_i) \arrow[u] & & \\ & & \mathcal{O}_{\tilde{X}} \arrow[u] & & \\ & & 0 \arrow[u] & & \end{tikzcd} \tag{1.7}$$

where $e_0=e_1=e_2=1$ and $e_3=e_4=2$. (1.6) implies $H^2(T_X)=0$ by (1.3). The above results yield, by the Riemann-Roch theorem, that

$$\dim H^1(T_X) = -\chi(T_X) = 10\chi(\mathcal{O}_X) - 2c_1^2 = 16. \qquad \text{Q.E.D.}$$

Lemma (1.5) asserts that the Kuranishi family

$$(1.8) \qquad \pi: \mathscr{X} \longrightarrow S \quad \text{with} \quad \iota: X \xrightarrow{\sim} X_{s_0} = \pi^{-1}(s_0) \qquad (s_0 \in S)$$

of the deformations of X is universal with a smooth parameter space S of dimension 16.

2. Effect of authomorphism

We continue to use the notations X, $\tilde{X}$, p, $\tilde{\tau}$ etc in the previous section.

(2.1) With the aid of the universal cover $\tilde{X}$ of weighted complete intersection, Oliverio ([O]) showed that the infinitesimal Torelli theorem holds for a general member X of the surfaces with $p_g=1$, $c_1^2=2$ and $\pi_1=\mathbf{Z}/2\mathbf{Z}$, i.e.

$$H^1(T_X) \longrightarrow \operatorname{Hom}(H^0(\Omega_X^2), H^1(\Omega_X^1))$$

is injective.

(2.2) On the other hand, the period map has 3-dimensional fibres through the surfaces X obtained from $\tilde{X}$ with $f^{(1)}(x_1, x_2)=g^{(1)}(x_1, x_2)=0$ (see (1.1)). We can prove this by an analogous argument of Todorov ([T.2]). That is, in this case, the bicanonical map of X

$$f_{|2K|}: X \longrightarrow \Sigma = f_{|2K|}(X) \subset \mathbf{P}^3$$

is a Galois cover with Galois group $(Z/2Z)^{\oplus 2}$ generated by the automorphisms σ_1 and σ_2 of X come from

$$\tilde{\sigma}_1(w, x_1, x_2, z_3, z_4) = (w, x_1, x_2, -z_3, z_4) \quad \text{and}$$
$$\tilde{\sigma}_2(w, x_1, x_2, z_3, z_4) = (w, x_1, x_2, z_3, -z_4)$$

respectively. Between X and Σ we have a quotient $X'=X/\langle\sigma_3\rangle$, where $\sigma_3=\sigma_1\sigma_2$, which is a $K3$ surface with 10 rational double points of type A_1. X is a double cover of X' whose branch locus is the pull-back of a hypersurface section on Σ plus 10 A_1's. The period map distinguishes $K3$ surfaces X' but does not distinguish the branch loci which contribute 3-dimension in the moduli space and appear as a fibre of the period map. (For a more precise description, see Remark 2.10 in [C.D].)

(2.3) We can explain the phenomenon (2.2) by a general viewpoint of an effect of an automorphism on a period map which we pointed out in [U.2]. This is the purpose in this section.

Let X, σ_3 and X' be as in (2.2). Set $\sigma=\sigma_3$ for simplicity. Note that in this case the canonical curve C is smooth because C is a fixed part by the invloution σ.

Lemma (2.4) (2.4.1) $\dim H^0(\Omega_X^1|C)^\sigma=\dim H^0(\Omega_X^1|C)=3$
(2.4.2) $\dim H^1(\Omega_X^1|C)^\sigma=1$
where $H^i(\Omega_X^1|C)^\sigma$ means (1)-eigen subspace by the induced action of σ.

Proof. Since $\sigma\,|\,C=\mathrm{id}_C$, the exact sequence

$$0\longrightarrow \check{N}_{C/X}\longrightarrow \Omega_X^1\,|\,C\longrightarrow \Omega_C^1\longrightarrow 0$$

splits and
$\check{N}_{C/X}$ is (-1)-eigen subbundle, and
Ω_C^1 is (1)-eigen subbundle.
Hence

$$H^i(\Omega_X^1|C)^\sigma \xrightarrow{\sim} H^i(\Omega_C^1) \qquad (i=0,\ 1).$$

This proves the lemma because $H^0(\check{N}_{C/X})=0$. QED.

Lemma (2.5) (2.5.1) $\dim H^2(X,\mathbf{Q})=20$.
(2.5.2) $\dim H^2(X,\mathbf{Q})^\sigma=12$.

Proof. (2.5.1) follows from

$$\chi_{\text{top}}(X) = 12\chi(\mathcal{O}_X) - c_1^2 = 12\cdot 2 - 2 = 22.$$

For (2.5.2), we use the following diagram:

$$(2.6)\qquad \begin{array}{ccc} X & \xleftarrow{\;p\;} & \hat{X} \\ \downarrow & & \downarrow{\scriptstyle q} \\ X':=X/\langle\sigma\rangle & \longleftarrow & \hat{X}':=\hat{X}/\langle\hat{\sigma}\rangle \end{array}$$

where p is the blowing-up of 10 isolated fixed points by σ which correspond to 10 A_1's on X', and $\hat{\sigma}$ is the induced involution on $\hat{X}$.

Then we have

$$p^*H^2(X,\mathbf{Q})^\sigma\oplus\Big(\bigoplus_{1\leqslant i\leqslant 10}\mathbf{Q}[E_i]\Big) = H^2(\hat{X},\mathbf{Q})^\sigma \xrightarrow{\sim} H^2(\hat{X}',\mathbf{Q}),$$

where $[E_i]$ $(1\leqslant i\leqslant 10)$ are the classes of exceptional curves with respect to p. Since $\hat{X}'$ is a minimal $K3$ surface, we have

$$\dim H^2(X,\mathbf{Q})^\sigma = 22 - 10 = 12. \qquad \text{Q.E.D.}$$

Lemma (2.7) (2.7.1) $\dim P^{2,0}(X)^\sigma=\dim H^0(\Omega_X^2)=1$.
(2.7.2) $\dim P^{1,1}(X)^\sigma=9$
where $P^{p,q}(X)$ is the primitive part of $H^{p,q}(X)$ with respect to the polarization by K_X.

Proof. In the diagram (2.6), the holomorphic 2-form on X comes from that on the $K3$-surface X'. This proves (2.7.1). (2.7.2) follows from (2.5.2) and (2.7.1) since the polarization (1, 1)-form is σ-invariant. Q.E.D.

Lemma (2.8) $\dim H^1(T_X)^\sigma = 12$.

Proof. From the exact sequence

$$0 \longrightarrow T_X \longrightarrow \Omega_X^1 \longrightarrow \Omega_X^1 \,|\, C \longrightarrow 0$$

and (1.5), we have

(2.9) $$0 \longrightarrow H^0(\Omega_X^1|C) \longrightarrow H^1(T_X) \xrightarrow{\psi} H^1(\Omega_X^1) \longrightarrow H^1(\Omega_X^1|C) \longrightarrow 0.$$

By virtue of (2.7.1), the homomorphisms in (2.9) are compatible with the induced actions of σ. Hence the restriction to the σ-invariant parts of (2.9) is also exact. Thus the lemma follows from (2.4) and (2.7.2). Q.E.D.

(2.10) Now we are ready to explain the phenomenon (2.2) as the effect of the automorphism σ on the period map.

Let X and σ be as above, and let $(\mathscr{X}, \pi, S, s_0, \iota)$ be the Kuranishi family (1.8) of the deformations of X. The universality of this family means

$$\mathrm{Aut}(\mathscr{X}, \pi, S, s_0) \xrightarrow[\sim]{\iota^*} \mathrm{Aut}(X), \quad \alpha \longmapsto \iota^{-1} \circ (\alpha | X_{s_0}) \circ \iota.$$

Denote

$H_{\mathbf{Z}} = P^2(X, \mathbf{Z}) =$ the 2^{nd} primitive cohomology proup of X with respect to the polarization by K_X.

$D = \left\{ F \,|\, 1\text{-dimensional subspace of } H_{\mathbf{C}} = H_{\mathbf{Z}} \otimes \mathbf{C} \text{ satisfying } \int_X F \wedge F = 0 \text{ and } \int_X F \wedge \bar{F} > 0 \right\}$

$\varphi \colon S \to D$ the period map of Griffiths ([Gri]).

$\mathrm{Aut}(X)$ acts on the Griffiths domain D through

$$\mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(H_{\mathbf{Z}}), \quad \alpha \longmapsto \alpha^{*-1}.$$

It is easy to see that the period map φ is $\mathrm{Aut}(X)$-equivariant with these induced actions on S and D. in particular, for $\sigma = \sigma_3 \in \mathrm{Aut}(X)$, we have the restriction of φ to the σ-fixed points:

(2.11) $$\varphi^\sigma \colon S^\sigma \longrightarrow D^\sigma.$$

Since σ is of finite order, S^σ and D^σ are smooth. Therefore

$$\dim_{s_0} S^\sigma = \dim H^1(T_X)^\sigma$$

$$\dim_{\varphi(s_0)} D^\sigma = \dim T_D(\varphi(s_0))^\sigma = \dim \mathrm{Hom}\,(P^{2,0}(X), P^{1,1}(X))^\sigma$$

Hence from (2.8) and (2.7), we have

$$\dim_{s_0} S^\sigma = 12$$
$$\dim_{\varphi(s_0)} D^\sigma = 9.$$

This implies

$$\dim_{s_0} (\varphi^\sigma)^{-1}\varphi^\sigma(s_0) \geqslant 12 - 9 = 3. \tag{2.12}$$

On the other hand, the morphism ψ in (2.9) is essentially the same as the differential of the period map φ at s_0 and hence

$$\dim_{s_0} \varphi^{-1}\varphi(s_0) \leqslant \dim H^0(\Omega_X^1|C) = 3 \tag{2.13}$$

by (2.4.1). Thus from (2.12) and (2.13) we can conclude that (2.11) is a smooth morphism of relative dimension 3. Q. E. D.

Remark (2.14) *With the aid of the universal cover $\tilde{X}$, which is a weighted complete intersection, we can classify all the automorphisms of the surfaces X with $p_g=1$, $c_1^2=2$ and $\pi_1=\mathbf{Z}/2\mathbf{Z}$ and K_X ample following the program in* [U.2], *and can examine the effect of each automorphism on the period map in the same way. (cf. Addendum of the present paper.)*

Remark (2.15) *We can also explain a similar phenomenon like* (2.2) *as an effect of an automorphism on the period map for the surfaces X with $p_g=1$, $c_1^2=2$ and $\pi_1=\{1\}$ and the bicanonical map*

$$f_{|2K|}: X \longrightarrow \Sigma = f_{|2K|}(X) \subset \mathbf{P}^3$$

is a double cover of a quadric surface Σ in $\mathbf{P}^3$. These are surfaces studied in [T.2].

3. Infinitesimal Torelli theorem by means of mixed Hodge structures

We continue to use the notations X, C, $\tilde{X}$, $\tilde{C}$, p and $\tilde{\tau}$ in the section 1.

By a similar method in [U.3], we can prove for X the infinitesimal Torelli theorem by means of the mixed Hodge structure on $X-C$:

Theorem (3.1) *Let X be a surface with $p_g=1$, $c_1^2=2$ and $\pi_1=\mathbf{Z}/2\mathbf{Z}$. Assume that the canonical cuve C is smooth and ample. Then the map*

$$\varphi: H^1(T_X(-\log C)) \longrightarrow \mathrm{Hom}\,(H^0(\Omega_X^2(\log C)), H^1(\Omega_X^1(\log C)))$$

defined by the contraction is surjective.

Proof. From the exact sequences

$$0 \longrightarrow T_X(-C) \longrightarrow T_X(-\log C) \longrightarrow T_C \longrightarrow 0 \quad \text{and}$$
$$0 \longrightarrow \Omega_X^p \longrightarrow \Omega_X^p(\log C) \longrightarrow \Omega_C^{p-1} \longrightarrow 0$$

(cf. [Ka], [D], [U.3]), we get the cohomology sequences:

$$\begin{aligned} &0 \longrightarrow H^1(T_X(-C)) \longrightarrow H^1(T_X(-\log C)) \xrightarrow{\gamma} H^1(T_C). \\ (3.2)\qquad &0 \longrightarrow H^0(\Omega_X^2) \xrightarrow{\alpha} H^0(\Omega_X^2(\log C)) \longrightarrow H^0(\Omega_C^1) \longrightarrow 0. \\ &0 \longrightarrow P^1(\Omega_X^1) \longrightarrow H^1(\Omega_X^1(\log C)) \xrightarrow{\beta} H^1(\mathcal{O}_C) \longrightarrow 0. \end{aligned}$$

Set

$$T_2 = \operatorname{Im}\{\gamma\colon H^1(T_X(-\log C)) \longrightarrow H^1(T_C)\},$$
$$T' = \{\theta \in \operatorname{Hom}(H^0(\Omega_X^2(\log C)), H^1(\Omega_X^1(\log C))) \mid \beta\theta\alpha = 0\} \text{ and}$$
$$T_1' = \{\theta \in T' \mid \beta\theta = 0\}.$$

Then, from (3.2), we get a commutative exact diagram:

$$(3.3)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & H^1(T_X(-C)) & \longrightarrow & H^1(T_X(-\log C)) & \longrightarrow & T_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi_2} & & \\ 0 & \longrightarrow & T_1' & \longrightarrow & T' & \longrightarrow & \operatorname{Hom}(H^0(\Omega_C^1), H^1(\mathcal{O}_C)) & \longrightarrow & 0. \end{array}$$

The injectivity of φ follows from the injectivity of φ_1 and φ_2 which we will prove in a sequence of lemmas.

Lemma (3.4) $\varphi_1'\colon H^1(T_X(-C)) \longrightarrow \operatorname{Hom}(H^0(\Omega_X^2(\log C)), H^1(\Omega_X^1))$ *is injective.*

Proof. Step 1. Under the identification

$$H^0(\Omega_{\tilde{X}}^2) = H^0(\mathcal{O}_{\tilde{X}}(1)) = \mathbf{C}[w, x_1, x_2, z_3, z_4]_1$$

the induced action of the involution $\tilde{\tau}$ is

$$\tilde{\tau}(w, x_1, x_2, z_3, z_4) = (w, -x_1, -x_2, -z_3, -z_4) \quad \text{(see (1.1))}.$$

Set

$$(3.5)\quad \begin{aligned} W = H^0(\Omega_{\tilde{X}}^2)^{(-1)} &= \text{the } (-1)\text{-eigen subspace of } H^0(\Omega_{\tilde{X}}^2) \text{ under } \tilde{\tau}\text{-action} \\ &= \mathbf{C}x_1 \oplus \mathbf{C}x_2, \end{aligned}$$

and consider the Koszul complex $K^\cdot$ defined by

$$(T_{\tilde{X}}(-\tilde{C}), \mathcal{O}_{\tilde{X}}(1), W) \quad (\text{resp. } (T_{\tilde{X}}, \mathcal{O}_{\tilde{X}}(1), W))$$

in [L.W.P], i.e.

$$K^p = (T_{\tilde{X}}(-\tilde{C}) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_{\tilde{X}}(1)^{\otimes p}) \otimes_{\mathbf{C}} \wedge^p W^\vee \quad (\text{resp. } = (T_{\tilde{X}} \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_{\tilde{X}}(1)^{\otimes p} \otimes_{\mathbf{C}} \wedge^p W^\vee)$$

and the spectral sequence of hypercohomology of this complex:

$$E_1^{p,q} = H^q(K^p).$$

Step 2. Since x_1, x_2 is a regular sequence in the canonical ring $\mathbf{C}[w, x_1, x_2, z_3, z_4]/(f, g)$, the Koszul complex $K^\cdot$ is exact in both cases. Hence, in paraticular, we have, in both cases,

$$E^1 = H^1(K^\cdot) = 0. \tag{3.6}$$

Step 3. On the other hand,

$$E_1^{2,0} = H^0(K^2) = H^0(\Omega_{\tilde{X}}^1)\otimes \wedge^2 W^\vee \quad (\text{resp. } = H^0(\Omega_{\tilde{X}}^1\otimes\mathcal{O}_{\tilde{X}}(1))\otimes \wedge^2 W^\vee),$$

where we use the identification $T_{\tilde{X}}\otimes\Omega_{\tilde{X}}^2=\Omega_{\tilde{X}}^1$. Since $\tilde{X}$ is simply connected, we have $H^0(\Omega_{\tilde{X}}^1)=0$. The vanishing of $H^0(\Omega_{\tilde{X}}^1\otimes\mathcal{O}_{\tilde{X}}(1))$ follows from the exact diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & \check{N}_{\tilde{X}/\mathbf{P}}\otimes\mathcal{O}_{\tilde{X}}(1) & \longrightarrow & \Omega_{\mathbf{P}}^1\otimes\mathcal{O}_{\tilde{X}}(1) & \longrightarrow & \Omega_{\tilde{X}}^1\otimes\mathcal{O}_{\tilde{X}}(1) & \longrightarrow & 0, \\
 & & & & \downarrow & & & & \\
 & & & & \bigoplus_{0\leq i\leq 4}\mathcal{O}_{\tilde{X}}(-e_i)\oplus\mathcal{O}_{\tilde{X}}(1) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathcal{O}_{\tilde{X}}(1) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

where $e_0=e_1=e_2=1$ and $e_3=e_4=2$. In fact, since $\oplus H^0(\oplus\,\mathcal{O}_{\tilde{X}}(1-e_i))\xrightarrow{\sim} H^0(\mathcal{O}_{\tilde{X}}(1))$, we see that $H^0(\Omega_{\mathbf{P}}^1\otimes\mathcal{O}_{\tilde{X}}(1))=0$. Since $\check{N}_{\tilde{X}/\mathbf{P}}\simeq\mathcal{O}_{\tilde{X}}(-4)^{\oplus 2}$, we see that $H^1(\check{N}_{\tilde{X}/\mathbf{P}}\otimes\mathcal{O}_{\tilde{X}}(1))=0$ (see [M]). Hence we get $H^0(\Omega_{\tilde{X}}^1\otimes\mathcal{O}_{\tilde{X}}(1))=0$. Therefore, we have in both cases,

$$E_1^{2,0} = H^0(K^2) = 0. \tag{3.7}$$

Step 4. From (3.6), (3.7) and the well-known exact sequence:

$$0 \longrightarrow E_2^{1,0} \longrightarrow E^1 \longrightarrow E_2^{0,1} \longrightarrow E_2^{2,0} \longrightarrow E^2,$$

we get, in both cases,

$$E_2^{0,1} = \operatorname{Ker}\{H^1(K^0) \longrightarrow H^1(K^1)\} = 0, \text{ i.e.}$$

$$\left.\begin{array}{l} H^1(T_{\tilde{X}}(-\tilde{C})) \longrightarrow H^1(T_{\tilde{X}})\otimes W^\vee \\ H^1(T_{\tilde{X}}) \longrightarrow H^1(\Omega_{\tilde{X}}^1)\otimes W^\vee \end{array}\right\} \text{ are injective.}$$

This implies the injectivity of the map

(3.8) $$H^1(T_{\tilde{X}}(-\tilde{C})) \longrightarrow H^1(\Omega^1_{\tilde{X}}) \otimes (W^\vee)^{\otimes 2}.$$

Step 5. Since the induced action of $\tilde{\tau}$ is the identity on $(W^\vee)^{\otimes 2}$, (3.8) implies the injectivity of the map $\tilde{\varphi}_1$ in the following commutative diagram:

(3.9)
$$\begin{array}{ccc} H^1(T_X(-C)) & \xrightarrow{\varphi_1'} & \mathrm{Hom}\,(H^0(\Omega^2_X(\log C)),\ H^1(\Omega^1_X)) \\ \downarrow & & \downarrow \\ H^1(T_{\tilde{X}}(-\tilde{C})) & \xrightarrow{\tilde{\varphi}_1} & \mathrm{Hom}\,(H^0(\Omega^2_{\tilde{X}}(\log \tilde{C}))^{\tilde{\tau}},\ H^1(\Omega^1_{\tilde{X}})) \end{array}$$

where $H^0(\Omega^2_{\tilde{X}}(\log \tilde{C}))^{\tilde{\tau}}$ stands for $\tilde{\tau}$-invariant subspace. Obviously, by (1.3), the virtical maps in (3.9) are injective. Thus φ_1' is injective. Q. E. D.

Lemma (3.10) $\varphi_1\colon H^1(T_X(-C)) \to T_1' \subset \mathrm{Hom}\,(H^0(\Omega^2_X(\log C)),\ H^1(\Omega^1_X(\log C)))$ is injective.

Proof. By the last exact sequence in (3.2), it is enough to show that

$$\mathrm{Im}\,\{H^1(T_X(-C)) \otimes H^0(\Omega^2_X(\log C)) \longrightarrow H^1(\Omega^1_X)\} \subset P^1(\Omega^1_X).$$

Let $\omega \in H^{1,1}(X)$ be the class of the canonical curve C. For $\theta \in H^1(T_X(-C))$ and $\xi \in H^0(\Omega^2_X(\log C))$, we have the formula of contraction:

$$\theta \cdot (\omega \wedge \xi) = (\theta \cdot \omega) \wedge \xi + \omega \wedge (\theta \cdot \xi).$$

Since $\omega \wedge \xi = 0$ and $\theta \cdot \omega = 0$ (in $H^2(\mathcal{O}_X(-C))$, we see $\omega \wedge (\theta \cdot \xi) = 0$ (in $H^{2,2}(X)$), i.e. $\theta \cdot \xi \in P^1(\Omega^1_X)$. Q. E. D.

Lemma (3.11) *Let Y be a smooth hyperelliptic curve of genus $g \geqslant 2$. Denote by T the subspace of $H^1(T_Y)$ consisting of the first order infinitesimal deformations of Y which are hyperelliptic. Then, the restriction to T of the initesimal period map is injective.*

Proof. This lemma must be well-known (e.g. an easy consequence of a great work [O.S]). But we will give a proof for the readers' convenience. Recall that a hyperelliptic curve Y of genus g can be represented as a double cover $Y \to \mathbf{P}^1$ ramified over $2g+2$ points, say $P_1, \ldots, P_{2g+2}$. Set $M = \mathcal{O}_{\mathbf{P}^1}(g+1)$ and denote by $s \in H^0(M^2)$ a global equation of the divisor $\sum_i P_i$. Then s gives an $\mathcal{O}_{\mathbf{P}^1}$-algebra structure on $\mathcal{O}_{\mathbf{P}^1} \oplus M^{-1}$ by

$$M^{-2} \xrightarrow{\otimes s} \mathcal{O}_{\mathbf{P}^1},$$

with which

$$Y = \mathcal{S}pec_{\mathbf{P}^1}\,(\mathcal{O}_{\mathbf{P}^1} \oplus M^{-1}).$$

It is easy to derive the following relations of cohomology groups (cf. [V]):

$$
(3.12)\qquad
\begin{aligned}
&H^1(T_Y) \xrightarrow{\sim} H^1(T_{\mathbf{P}^1}(-\log(\sum_i P_i))) \oplus H^1(T_{\mathbf{P}^1} \otimes M^{-1})\\
&H^0(\Omega_Y^1) \xrightarrow{\sim} H^0(\Omega_{\mathbf{P}^1}^1) \oplus H^0(\Omega_{\mathbf{P}^1}^1(\log(\sum P_i)) \otimes M^{-1})\\
&H^1(\mathcal{O}_Y) \xrightarrow{\sim} H^1(\mathcal{O}_{\mathbf{P}^1}) \oplus H^1(M^{-1})
\end{aligned}
$$

The first (resp. second) terms in the right hand sides of (3.12) are (1)-(resp. (-1)-) eigen subspaces by the covering transformation. Hence

$$T = H^1(T_{\mathbf{P}^1}(-\log(\sum P_i)))$$

and our map is

$$(3.13)\quad H^1(T_{\mathbf{P}^1}(-\log(\sum P_i))) \longrightarrow \mathrm{Hom}\,(H^0(\Omega_{\mathbf{P}^1}^1(\log(\sum P_i)) \otimes M^{-1}),\ H^1(M^{-1})).$$

The dual of this map, i.e. the codifferential, is

$$H^0((\Omega_{\mathbf{P}^1}^1 \otimes M)^{\otimes 2}) \longleftarrow H^0(\Omega_{\mathbf{P}^1}^1 \otimes M)^{\otimes 2}.$$

This is obviously surjective, since $\Omega_{\mathbf{P}^1}^1 \otimes M \simeq \mathcal{O}_{\mathbf{P}^1}(g-1)$. Therefore (3.13) is injective. Q.E.D.

Lemma (3.11) together with (1.4) yields the injectivity of φ_2. This together with Lemma (3.10) completes the proof of Theorem (3.1).

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